A warm-start approach for large-scale stochastic linear programs

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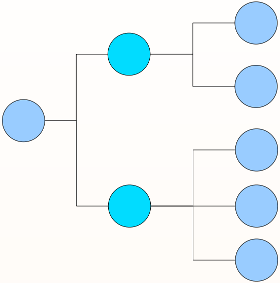
What is stochastic programming

- Model uncertainty through the analysis of possible future scenarios
- Alternating sequence of decisions and random realisations
- Robust decision making
- And much more!

Event tree



To each node of the tree we associate:

- a set of constraints
- an objective function
- the conditional probability of visit from the parent node

Notation

t stage

l_t index of a node of stage t

$a(l_t)$ ancestor of node l_t

n^{l_t} node data: $\{T^{l_t}, W^{l_t}, h^{l_t}, q^{l_t}, p^{l_t}\}$

Model of the dynamics of the system (at node l_t):

$$\begin{aligned} \min \quad & \sum_{l_t} p^{l_t} (q^{l_t})^T x^{l_t} \\ \text{s.t.} \quad & T^{l_t} x^{a(l_t)} + W^{l_t} x^{l_t} = h^{l_t} \\ & x^{l_t} \geq 0 \end{aligned}$$

Complete deterministic equivalent formulation

$$\begin{array}{llcll}
\min & (q^{l_1})^T x^{l_1} & + \displaystyle\sum_{l_2=L_1+1}^{L_2} p^{l_2}(q^{l_2})^T x^{l_2} + \ldots + \displaystyle\sum_{l_T=L_{T-1}+1}^{L_T} p^{l_T}(q^{l_T})^T x^{l_T} & & \\
\text{s.t.} & & W^{l_1}x^{l_1} = h^{l_1}, & & \\
& T^{l_2}x^1 + & W^{l_2}x^{l_2} = h^{l_2}, & & l_2 = L_1+1,\ldots,L_2, \\
& \vdots & \vdots & & \\
& T^{l_T}x^{a(l_T)} + & W^{l_T}x^{l_T} = h^{l_T}, & & l_T = L_{T-1}+1,\ldots,L_T, \\
& & x^{l_t} \geq 0, & & l_t = 1,\ldots,L_T.
\end{array}$$

The SMPS format

Standard formulation of multistage stochastic programs.

A problem in SMPS format is defined through 3 text files:

Core file: underlying deterministic problem in MPS format;

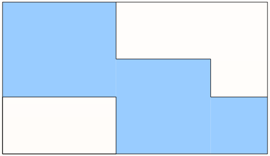
Time file: information about the breaking up in stages;

Stoch file: list of variations to the core data for each scenario.

Provides all information about the structure of the problem.

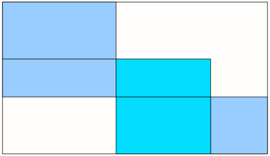
The SMPS files

Core



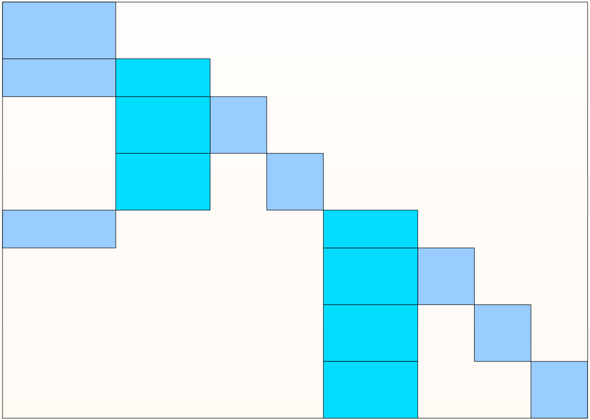
The SMPS files

Core + Time

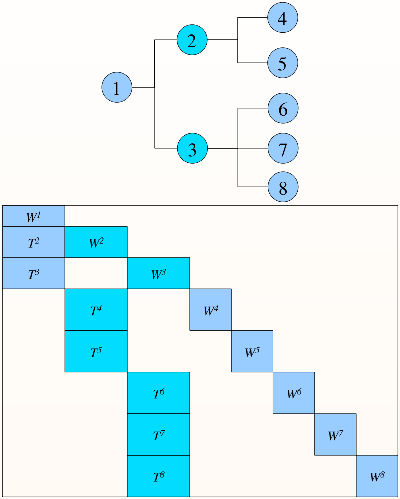


The SMPS files

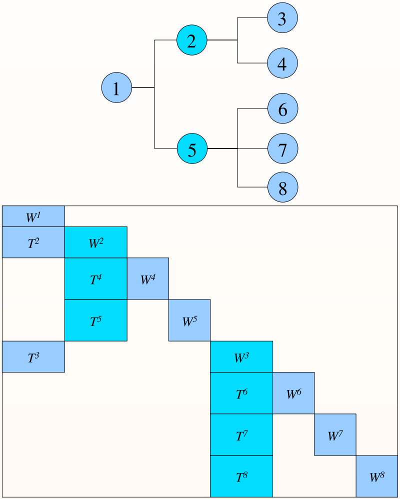
Core + Time + Stoch



Structure of the deterministic equivalent



Breadth-first ordering



Depth-first ordering

Structure of the deterministic equivalent

$$
\begin{array}{cccccccc}
q^1 & q^2 & q^3 & q^4 & q^5 & q^6 & q^7 & q^8
\end{array}
$$

$$
\begin{bmatrix}
W^1 & & & & & & & \\
T^2 & W^2 & & & & & & \\
T^3 & & W^3 & & & & & \\
 & T^4 & & W^4 & & & & \\
 & T^5 & & & W^5 & & & \\
 & & T^6 & & & W^6 & & \\
 & & T^7 & & & & W^7 & \\
 & & T^8 & & & & & W^8
\end{bmatrix}
\qquad
\begin{bmatrix}
h^1 \\ h^2 \\ h^3 \\ h^4 \\ h^5 \\ h^6 \\ h^7 \\ h^8
\end{bmatrix}
$$

Issues with the deterministic equivalent approach

The deterministic equivalent formulation produces problems of extremely large size, even when starting from a small core.

Example: fxm	rows	cols	nonzeros
Core matrix	330	457	2,566
3 stages, 6 nodes:	6,200	9,492	54,589
4 stages, 16 nodes:	386,940	517,282	4,518,039

- A detailed description produces robust decisions
- Detailed event trees can be very large
- The dimensions involved explode

The way forward

Enter interior point methods:

- IPM solvers are available in the community (CPLEX Barrier, PCx, HOPDM, etc.)
- Competitiveness of IPMs grows with the problem size
- Parallel implementations are possible

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And:

- Structure-exploiting (parallel) software (OOPS)
- Exploiting the stochastic structure to warm-start the IPMs

Linear programming and optimality conditions

Linear programming problem

$$\begin{array}{rl} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array} \qquad \begin{array}{rl} \max & b^T y \\ \text{s.t.} & A^T y + s = c \\ & s \geq 0 \end{array}$$

Karush-Kuhn-Tucker (KKT) conditions for optimality

$$\begin{aligned} Ax - b &= 0 \\ A^T y + s - c &= 0 \\ \forall i: \; x_i s_i &= 0 \\ x, s &\geq 0 \end{aligned} \qquad F(x,y,s) = \begin{bmatrix} Ax - b \\ A^T y + s - c \\ XSe \end{bmatrix} = 0, \quad x, s \geq 0$$

Derivation of path-following methods

Perturb the complementarity conditions:

$$XSe = \mu e$$

IPMs solve a sequence of problems parametrised by μ.

Let $\mu \to 0$:

- The perturbed conditions better approximate the original KKT conditions
- The solution traces a continuous path from the starting point to the optimal solution (central path)

Primal–dual interior point methods

Basic structure of an IPM iteration

- Given an iterate (x, y, s) for which $(x, s) > 0$
- Solve the perturbed KKT conditions with Newton's method

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta s \end{bmatrix} = \begin{bmatrix} b - Ax \\ c - A^T y - s \\ -XSe + \mu e \end{bmatrix}$$

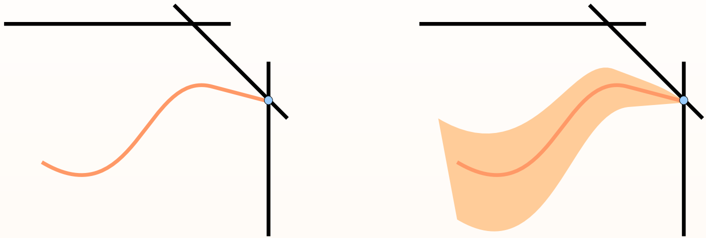
- Move to the next point with stepsize α such that

$$(x + \alpha \Delta x, s + \alpha \Delta s) > 0$$

- Here A is the whole deterministic equivalent

Centrality

IPMs follow the central path to find the optimal solution.
The iterates lie in some neighbourhood of the central path.

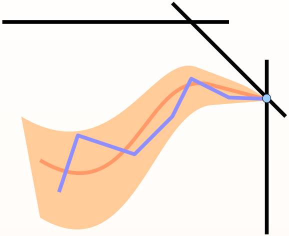


Polynomial complexity:

in theory: $\mathcal{O}(\sqrt{n})$ or $\mathcal{O}(n)$ iterations

in practice: $\mathcal{O}(\ln n)$ iterations

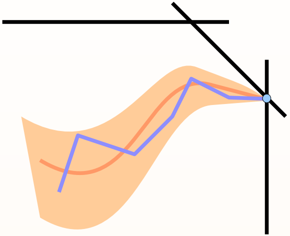
Good behaviour and bad behaviour



Good:

- central starting point
- remain in the neighbourhood of the central path in all iterations

Good behaviour and bad behaviour



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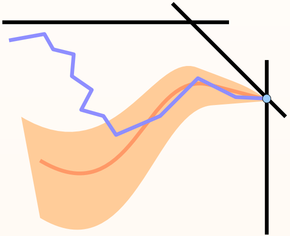


Bad:

- iterate close to the boundary
- many iterations spent in retrieving centrality before converging

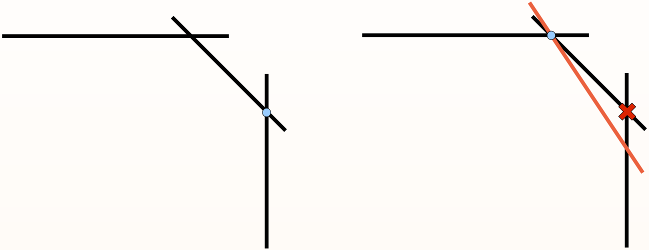
Warm-start strategies

A warm-start strategy uses the solution to a problem instance to initialise the next problem.

- Important if we are solving a sequence of problems
- Often we may expect that the solution to one problem is close to the solution of the next
- An advanced starting point may lead to reduced computational time than solving the problem from scratch

Warm-start with the simplex method

The solution of a problem is a vertex:



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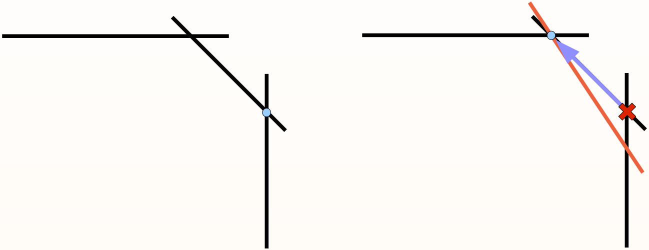
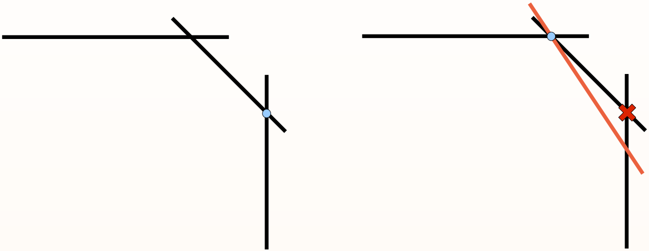


- ideal starting point for the modified instance
- optimality recovered in a few iterations (if there are not too many changes in the problems)

Warm-start with interior point methods

The solution of a problem is arbitrarily close to a vertex:



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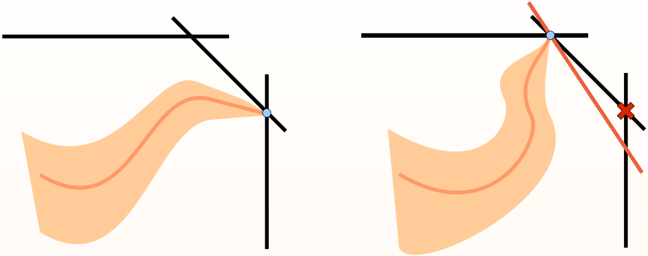

- worst possible starting point

Warm-start with interior point methods

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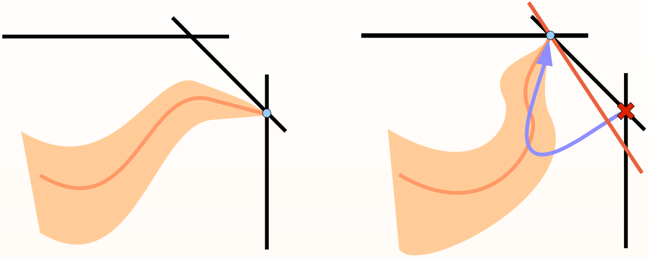

- worst possible starting point
- some iterations to recover centrality in the new central path
- some iterations for optimality

Warm-start issues with IPMs

Contradictory requirements:

- Point should be close to the solution
- Point should be away from the boundary

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Current attempts:

- Store an "advanced" iterate (3–4 digits of accuracy)
- Take special care of centrality
- Restore primal and dual feasibility with independent directions

Assumptions and setup

Main assumption:

- No knowledge on the underlying stochastic processes
- An event tree is given

Problem setup:

- Required to solve an instance with a specific tree
- Stochastic problems are given in the SMPS format
- We generate and solve the deterministic equivalent

Reduced event tree

Observation:

Very detailed event trees provide a fine-grained solution to a problem that could have been solved more coarsely with a much smaller tree.

Idea:

Use the solution to a smaller instance of the problem to generate a warm-start point.

Reduced tree generation

Two complementary strategies:

1. Span the breadth of the tree
    - Choose some of the nodes at stage k (where k is small)
    - Choose all their ancestors up to the root node

Reduced tree generation

Two complementary strategies:

1. Span the breadth of the tree
   - Choose some of the nodes at stage k (where k is small)
   - Choose all their ancestors up to the root node
2. Choose the most representative scenario in each subtree
   - Define a "scenario distance"
   - Minimize the distance from an average scenario

Scenario distance and representative scenarios

Distance between two nodes at period t:

$$d(n^{i_t}, n^{j_t}) = \|T^{i_t} - T^{j_t}\| + \|W^{i_t} - W^{j_t}\| + \|h^{i_t} - h^{j_t}\| + \|q^{i_t} - q^{j_t}\|$$

Scenario distance and representative scenarios

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Distance between two scenarios:

$$D(s_i, s_j) = \sum_{t=1}^{T} d(n^{i_t}, n^{j_t}), \quad i_t \in s_i,\ j_t \in s_j$$

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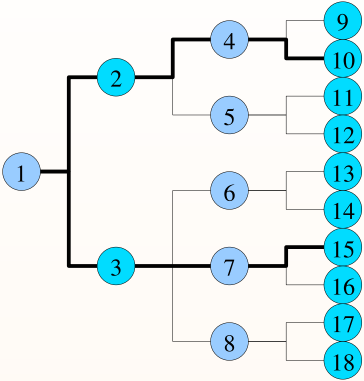
$$D(s_i, s_j) = \sum_{t=1}^{T} d(n^{i_t}, n^{j_t}), \quad i_t \in s_i,\ j_t \in s_j$$

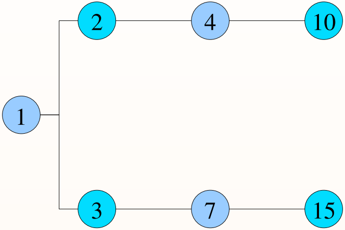
In each subtree of the given tree, the representative scenario s^* is the one that minimizes the weighted distance from an average scenario $\bar{s}$

$$s^* = s_k, \quad k = \arg\min_{i \in S} (1 - p_i) D(s_i, \bar{s})$$

Scenario reduction



Complete tree



Reduced tree

Main steps of the algorithm

Exploit the structure of the stochastic program

1. Find a reduced event tree

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Good news: The reduced problem is very easy to solve!

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Good news: The reduced problem is very easy to solve!

Bad news: The dimensions of the two problems don't match...

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2. Solve the reduced deterministic equivalent with loose accuracy
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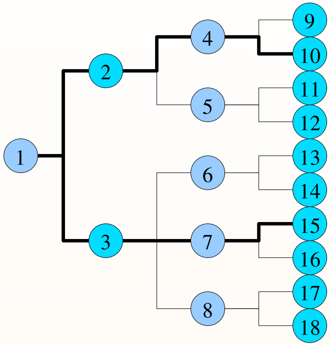
Good news: The reduced problem is very easy to solve!

Bad news: The dimensions of the two problems don't match. . .

Good news: We can exploit the structure again!

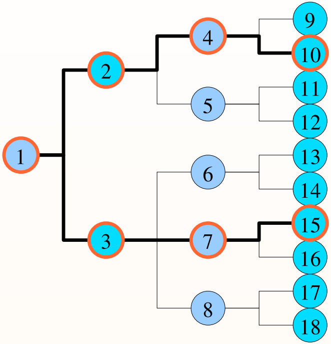
Construction of the warm-start iterate



1	2	3	4			7			10					15			

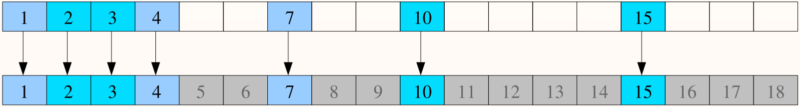
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18

Construction of the warm-start iterate



Nodes in the reduced tree:

the solution is already available



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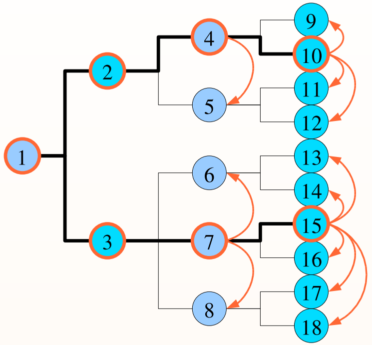
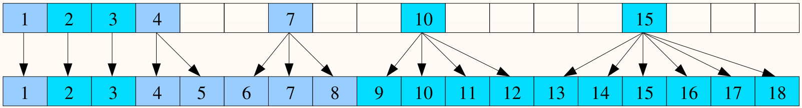


Nodes in the reduced tree:
the solution is already available

Remaining nodes:
copy the solution from the corresponding reduced-tree node



Numerical results I

Collection of standard SMPS problems solved with HOPDM:

- 2 scenarios in the reduced tree
- Reduced problem optimality tolerance: 5.0×10^{-1}
- Complete problem optimality tolerance: 5.0×10^{-8}

Numerical results with HOPDM

Problem data			Cold start		Warm start	
Name	Stgs	Scens	Iters	Seconds	Iters	Seconds
fxm2-16	2	16	22	1.2	13	1.0
fxm3-6	3	36	30	1.5	17	1.3
fxm3-16	3	256	40	31.1	20	20.7
fxm4-6	4	216	30	8.2	22	8.3
fxm4-16	4	4096	41	218.3	27	182.6
pltexpA3-16	3	256	26	153.8	14	87.8
pltexpA4-6	4	216	36	55.8	16	27.5
pltexpA5-6	5	1296	81	772.0	30	311.5
storm27	2	27	41	95.4	22	53.2
storm125	2	125	73	107.3	36	69.1
storm1000	2	1000	107	1498.3	45	831.5

Capacity assignment problem with uncertain demand

$$\min_x \; E_d[f(x,d)] \quad \text{s.t.} \quad \sum_{l\in\mathcal{A}} c_l x_l \le M, \;\; x \ge 0,$$

$$\begin{aligned} f(x,d) = \min \quad & \sum_{k\in\mathcal{D}} (d_k - \sum_{p\in P_k} z_p) \\ \text{s.t.} \quad & \sum_{k\in\mathcal{D}} \sum_{p\in\mathcal{P}_k : l\in p} z_p \le x_l \quad && \forall l \in \mathcal{A} \\ & \sum_{p\in\mathcal{P}_k} z_p \le d_k && \forall k \in \mathcal{D} \\ & z_p \ge 0 \end{aligned}$$

Numerical results II

Problems formulated as SMPS and solved with OOPS:

- 2 scenarios in the reduced tree (serial) or 4 scenarios (parallel)
- Reduced problem optimality tolerance: 5.0×10^{-1}
- Complete problem optimality tolerance: 5.0×10^{-4}

Numerical results with OOPS (serial)

Problem data			Cold start		Warm start	
Name	Stgs	Scens	Iters	Seconds	Iters	Seconds
mnx-200	2	200	13	12.9	7	7.3
mnx-800	2	800	17	58.8	10	39.5
mnx-1600	2	1600	19	131.1	10	68.8
jlg-200	2	200	45	164.9	17	39.5
jlg-800	2	800	27	353.4	10	152.9
jlg-1600	2	1600	32	855.3	13	360.6
mgntA-100	2	100	28	260.0	14	156.2
mgntA-200	2	200	50	877.1	35	690.6
mgntA-400	2	400	40	1470.3	14	572.5
mgntB-100	2	100	23	511.1	14	318.0
mgntB-200	2	200	25	909.4	8	332.4
mgntB-400	2	400	29	2154.5	7	538.1

Numerical results with OOPS (parallel)

Problem data			Cold start		Warm start	
Name	Stgs	Scens	Iters	Seconds	Iters	Seconds
mnx-200	2	200	13	4.6	7	3.5
mnx-800	2	800	17	18.8	10	10.7
mnx-1600	2	1600	19	50.3	10	31.4
jlg-200	2	200	45	49.9	17	20.7
jlg-800	2	800	29	130.5	10	50.1
jlg-1600	2	1600	35	286.1	14	129.7
mgntA-100	2	100	28	76.9	14	51.6
mgntA-200	2	200	50	256.4	34	195.3
mgntA-400	2	400	40	410.9	14	181.6
mgntB-100	2	100	23	137.5	14	103.9
mgntB-200	2	200	25	284.2	8	140.5
mgntB-400	2	400	29	605.5	7	211.6

Conclusions

- Reduced tree solutions contain valuable information to construct a good warm-start iterate
- In this case, interior point methods can be used successfully in warm-start situations
- Exploiting the structure gives once again an additional advantage