A computational experience with search directions in Interior Point Methods for Linear Programming

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Linear programming and optimality conditions

- Linear programming problem and KKT conditions

$$\begin{array}{rl} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array} \qquad \begin{bmatrix} Ax - b \\ A^T y + s - c \\ XSe \end{bmatrix} = 0 \quad x, s \geq 0$$

- Perturb the complementarity conditions

$$XSe = \mu e$$

- Solve the perturbed KKT conditions with Newton's method

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta s \end{bmatrix} = \begin{bmatrix} b - Ax \\ c - A^T y - s \\ -XSe + \mu e \end{bmatrix}$$

Mehrotra's algorithm: Predictor direction

- Exploit linearity: solve independently for two right-hand sides

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta s \end{bmatrix} = \underbrace{\begin{bmatrix} b - Ax \\ c - A^T y - s \\ -XSe \end{bmatrix}}_{predictor} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \mu e \end{bmatrix}}_{corrector}$$

- Predictor: set $\mu = 0$ and solve for the direction Δ_a
- Evaluate the allowed stepsizes

$$\alpha_p = \max_{\alpha} : x + \alpha \Delta_a x \geq 0 \qquad \alpha_d = \max_{\alpha} : s + \alpha \Delta_a s \geq 0$$

- Predicted gap: $g_a = (x + \alpha_p \Delta_a x)^T (s + \alpha_d \Delta_a s)$
- Use the predicted gap to estimate the centering term

$$\mu = \left(\frac{g_a}{x^T s} \right)^3 \frac{x^T s}{n}$$

Mehrotra's algorithm: Corrector direction

- Error in taking a full step in the predictor

$$(X+\Delta_a X)(S+\Delta_a S)e = XSe + \underbrace{(S\Delta_a x + X\Delta_a s)}_{-XSe} + \Delta_a X \Delta_a S e$$

- Consider a second order term and solve for Δ_c

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \mu e - \Delta_a X \Delta_a S e \end{bmatrix}$$

- Find the stepsizes in the combined direction $\Delta = \Delta_a + \Delta_c$
- Usually the stepsizes are much better than those obtained by the predictor

Multiple centrality correctors

- Try to increase the stepsizes: $\tilde{\alpha} = \min(\alpha + \delta, 1)$
- Move the trial point in the neighbourhood

$$\tilde{v}_i = x_i(\tilde{\alpha}) s_i(\tilde{\alpha}) \in \mathcal{N}_s(\gamma) = \{x_i s_i : \gamma\mu \le x_i s_i \le \frac{1}{\gamma}\mu\}$$

- Define an achievable target

$$t_i = \begin{cases} 0 & \text{if } \tilde{v}_i \in [\gamma\mu, \frac{1}{\gamma}\mu] \\ \gamma\mu - \tilde{v}_i & \text{if } \tilde{v}_i < \gamma\mu \\ \frac{1}{\gamma}\mu - \tilde{v}_i & \text{if } \tilde{v}_i > \frac{1}{\gamma}\mu \end{cases} \qquad rhs = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}$$

- The number of correctors allowed depends on

$$\frac{\textit{Factorization effort}}{\textit{Backsolve effort}}$$

- A corrector is accepted if $\hat{\alpha} \ge \alpha + \rho\delta$

Practical pitfalls

- The stepsizes in the affine-scaling direction can be very short, especially if the point is badly centered
- Mehrotra's corrector is computed on the basis of full step in affine scaling direction
- Sometimes the magnitude of the corrector is much larger than the magnitude of the predictor (Cartis, 2005)
- Short steps in the combined direction may be produced

Weighting the corrector direction

- Cartis (2005) suggests weighting the corrector by α^2, based on a quadratic approximation of a local path from the current point to a target on the central path (PDSOC)
- Generalize Mehrotra's scheme

$$\Delta^{\omega} = \Delta_a + \omega \Delta_c$$

- Salahi, Peng and Terlaky (2005) propose $\omega = \alpha$ and safeguards based on centering
- Computational study on the weight of the corrector

Finding the best weight

- Find the corrector direction Δ_c

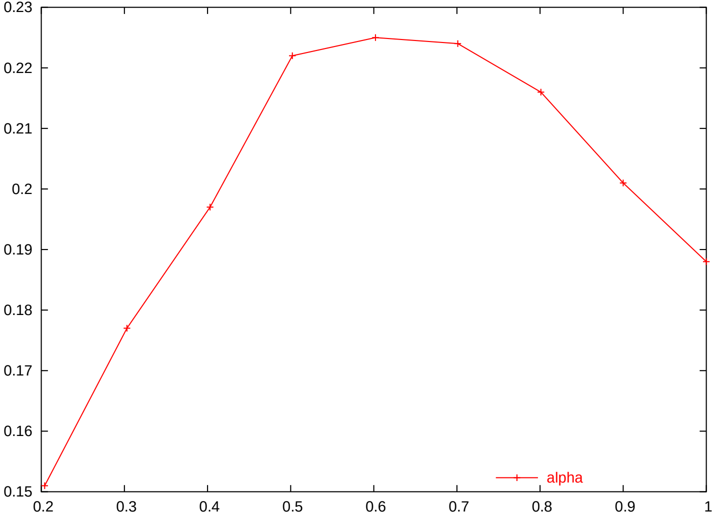
$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \mu e - \Delta_a X \Delta_a S e \end{bmatrix}$$

- Set $\omega \in [\alpha_p \alpha_d, 1]$ and compute

$$\Delta^\omega = \Delta_a + \omega \Delta_c$$

- Do a linesearch to find the optimal $\hat{\omega}$ that maximises the product of the stepsizes $\alpha_p^\omega \alpha_d^\omega$ in Δ^ω

Omega and alpha



Computational experience

- Results from a collection of 100 problems (Netlib and Kennington)
- Initial comparison between PCx code (Czyzyk, Mehrotra, Wright) and HOPDM code (Gondzio)
- Different linear algebra in PCx and HOPDM accounts for different choices concerning multiple centrality correctors
- Difference in termination criteria: implemented in HOPDM the criteria used in PCx
- Analysis done on number of iterations and number of backsolves

Computational results I

	PCx	HOPDM	HOPDM-ω	Change
Iterations	2086	1808	1749	-3%
Backsolves	5542	5547	5789	+5%
Backsolves/iter.	2.66	3.07	3.31	+8%

- Decrease in iteration count but increase in backsolves
- We are accepting more multiple centrality correctors than normally we would do
- Ask for more stepsize increase in multiple centrality correctors

Computational results II

- Use more aggressive centrality correctors

$$\hat{\delta} = 3\delta$$

	HOPDM+	HOPDM-ω+	Change	Total
Iterations	1765	1613	-8%	-10%
Backsolves	5132	5286	+3%	-4%
Backsolves/iter.	2.91	3.28	+13%	+7%

Future work

- Clarify the usefulness of ω
- Are there heuristics to (approximately) localise $\hat{\omega}$?
- Choice of settings in multiple centrality correctors